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A general method for determining upstream effects in stratified flow of finite depth over long two-dimensional obstacles

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A procedure for determining the flow which results from introducing a long two-dimensional obstacle of finite height into a unidirectional, two-dimensional, stable, but otherwise arbitrary stratified shear flow of finite depth is described. The method is based on a generalization of the known results for two-layer flows, described in Baines (1984). The flow is assumed to be hydrostatic with negligible mixing, and the stratified flow is represented by an arbitrary number of discrete layers, so that the model is hydraulic in character. The procedure involves the calculation of the changes to the steady-state flow resulting from successive increases in the height of the topography from zero. For a given initial flow, introduction of an obstacle only alters the flow in its vicinity for obstacle heights h_m less than a height h_c, where the flow is critical (implying zero wave speed) at the obstacle crest for some particular internal wave mode. Increasing the obstacle height further causes the flow to adjust to maintain a critical condition at the obstacle crest, and this causes disturbances with the structure of the critical mode to be propagated upstream. These may take the form of an upstream hydraulic jump or of a time-dependent rarefaction (implying a disturbance which becomes increasingly spread out with time), or both, depending on the nonlinear dispersive properties of the system. Their passage past a given upstream location results in a permanent change to the local velocity and density profiles. As the obstacle height is further increased these processes will continue until the flow becomes critical just upstream of the obstacle, or a fluid layer becomes blocked. For greater obstacle heights the above phenomena may be repeated with other modes. A numerical procedure which implements these processes has been developed, and examples of applicatons to two- and three-layer systems are given.

1. Introduction

This paper is concerned with the nature of stratified flow over isolated long two-dimensional obstacles of finite height in finite-depth systems. We formulate the problem in the following way; for a given stratified shear flow with arbitrary (but stable) velocity and density profiles, what is the nature of the flow that results from the introduction of an obstacle of given height and shape? The situation is illustrated in figure 1. The chief difficulty with this problem is that the introduction of the obstacle may cause the generation of disturbances which propagate arbitrarily far upstream and (in an inviscid system) alter the character of the incident flow. This means that, unless these disturbances can be determined, it is not possible to obtain the resulting flow field from a steady-state calculation. Furthermore, the generation

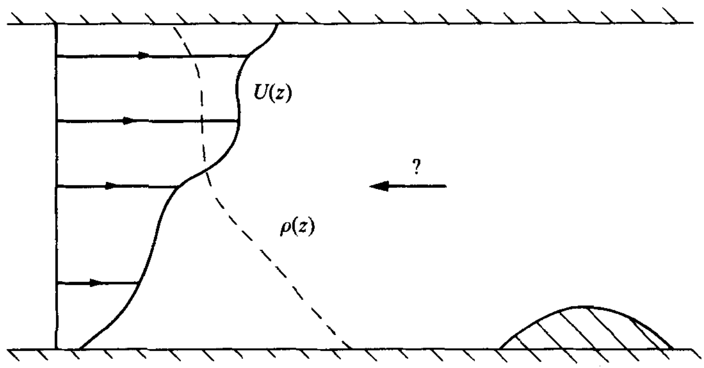

FIGURE 1. The introduction of an obstacle of finite height into a stratified shear flow may cause disturbances to be propagated arbitrarily far upstream which alter the oncoming flow.

process for these disturbances must be nonlinear in character, which means that this computation is not a simple matter.

The overall problem has been discussed in more detail in §1 of Baines (1984). That paper describes a theoretical and experimental study of two-layer flow, where only one internal mode is present. A reasonably complete picture of flow over obstacles for this relatively simple system was given by first constructing a theoretical model based on unifying the work of previous authors and then comparing the results with laboratory experiments. The success of this model led to the suggestion that it could be applied in modified form to more general velocity and density profiles, where many modes may be present. The development of such a general procedure for practical computations is the subject of this article.

The model considered here has the following properties or restrictions: (i) the flow is assumed to be two-dimensional, inviscid (apart from dissipation in jumps), and hydrostatic (implying sufficiently long topography); (ii) the topography has a single maximum height; (iii) the stratified shear flow may be approximated by an arbitrary but finite number of discrete homogeneous layers, which do not mix; (iv) the flow is initially two-dimensional and unidirectional, and (v) the fluid is bounded above by a rigid boundary or an infinitely deep homogeneous layer.

Scale analysis of the complete equations of motion indicates that as the horizontal gradient of the topography becomes small the flow becomes hydrostatic (excluding inside hydraulic jumps). However, recent work by Pratt (1984) has shown that this may not be so between two obstacles of comparable height, where cnoidal-type wavetrains may be present. The above restriction to a single topographic maximum is therefore required in order to guarantee hydrostatic flow.

The assumption that the fluid be layered has the interesting effect that critical layers are excluded from the flow. Horizontally propagating modes in layered flows may be speeds which are greater, equal to, or less than the flow velocity of any given layer. Critical layers, therefore, present no problem for the model. The question of whether or not the model lacks some relevant physics regarding critical layers is circumvented in the present paper by requiring that the initial stratified shear flow contain flow in one direction only (relative to the topography). The subsequent production of upstream disturbances by topography does not alter this condition upstream.

Restriction (v) above implies that wave energy may not escape upwards out of the

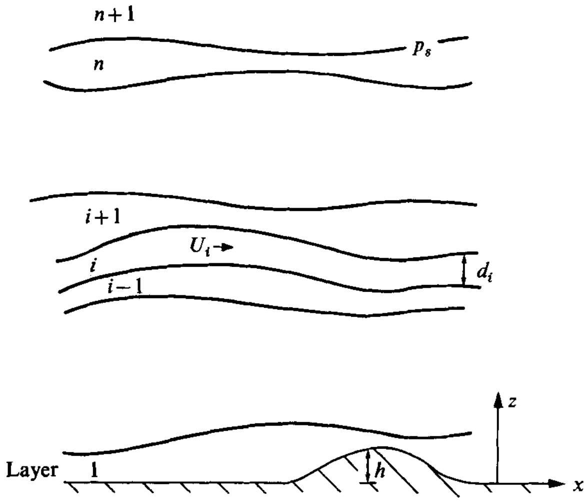

FIGURE 2. The configuration of layers for the model.

system, so that small-amplitude motions may be described in terms of discrete horizontally propagating modes. This is what is meant by a 'finite-depth' system. The relative properties of finite and infinite-depth systems are described in the review by Baines (1987).

The general method described in this paper is based on an extension of ideas which have been empirically validated for two-layer flows. It depends heavily on the work of previous authors, which it brings together to provide a unified physical picture. Numerical algorithms, in particular, are obtained from the work of Su (1976) and Lee & Su (1977).

The plan of the paper is as follows. The equations for layered flows are given in §2, and the equations governing linear wave disturbances on steady layered shear flows are given in §3. Hydraulic jumps and the equations governing them are summarized in §4. With these three sections as preliminaries, the general procedures for computing topographic effects in stratified shear flows is outlined in §5, with emphasis on the physical basis. Two examples to illustrate the method and the typical character of the results are given in §6, and the conclusions are summarized in §7.

2. Equations for layered flows

The equations governing the hydrostatic motion of n-incompressible layers of fluid may be expressed as (e.g. Lee & Su 1977)

$$\frac{\partial d_i}{\partial t}+\frac{\partial}{\partial x}(u_i d_i) = 0, \tag{2.1}$$

$$\frac{\partial u_i}{\partial t}+u_i\frac{\partial u_i}{\partial x} = -\frac{1}{\rho_i}\frac{\overline{\partial p}}{\partial x} \quad (i = 1,2,3,\dots,n), \tag{2.2}$$

where (see figure 2) u_i, d_i and ρ_i denote the velocity, thickness, and density of the ith layer respectively, x and t denote horizontal coordinate and time, and the overbar denotes a vertical average in the ith layer. u_i is assumed to be independent of z in each layer. If the pressure at the top of the nth layer is denoted by p_s, then for hydrostatic flow the pressure at a point within the ith layer is given by

$$p(x,z) = p_s + g \sum_{j=i+1}^{n} \rho_j d_j + g\rho_i \left(\sum_{j=0}^{i} d_j - z\right), \tag{2.3}$$

where z is the vertical coordinate and $d_0 = h(x)$ denotes topography on the bottom level surface $z = 0$. Equations (2.2) and (2.3) together give

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x}\left[\tfrac{1}{2}u_i^2 + g\sum_{j=0}^{i} d_j + g\sum_{j=i+1}^{n}\frac{\rho_j}{\rho_i}d_j + \frac{p_s}{\rho_i}\right] = 0 \quad (i = 1, \ldots, n). \tag{2.4}$$

Following Lee & Su we may accommodate two types of upper boundary: (i) a rigid surface at the top of the channel, so that

$$\sum_{j=0}^{n} d_j = D = \text{constant}, \tag{2.5}$$

and (ii) a free suface, where the nth layer is surmounted by a deep layer of density ρ_{n+1}. The hydrostatic pressure p_s is then given by

$$p_s = P_s - g\rho_{n+1}\left(\sum_{j=0}^{n} d_j - D\right), \tag{2.6}$$

where P_s is the value of p_s far upstream, and

$$D = \sum_{j=0}^{n} D_j, \tag{2.7}$$

where D_j is the value of d_j far upstream.

If we assume that the flow is steady and that $u_i = U_i$ far upstream, then (2.1), (2.4) integrate to

$$d_i u_i = D_i U_i, \tag{2.8}$$

$$\tfrac{1}{2}(u_i^2 - U_i^2) + g\sum_{j=0}^{n} \rho_{ij}(d_j - D_j) + \frac{1}{\rho_i}(p_s - P_s) = 0, \tag{2.9}$$

where

$$\left.\begin{aligned} \rho_{ij} &= 1 \qquad (j \leqslant i), \\ &= \rho_j/\rho_i \quad (j > i). \end{aligned}\right\} \tag{2.10}$$

Lee & Su show that, in general for a given stratified flow specified by upstream profiles U_i, D_i, $i = 1, \ldots, n$, there is a maximum obstacle height which may exist in this flow. In the remainder of this paper we present a method for obtaining the flow fields caused by obstacles whose height exceeds this value.

3. Linear disturbances in layered flows

Expressions for the celerity of linear disturbances to hydrostatic layered flow have been derived by Benton (1954) by finding extrema for energy and momentum transfer. Here we derive the same wave speeds and the associated wave structure directly from the equations of §2. Writing

$$\left.\begin{aligned} p_s &= P_s + p_s', \\ u_i &= U_i + u_i'(x,t), \\ d_i &= D_i + d_i'(x,t), \end{aligned}\right\} \tag{3.1}$$

where p_s', u_i', d_i' denote small perturbations, we obtain to lowest order

$$\frac{\partial u_i'}{\partial t}+\frac{\partial}{\partial x}\left[U_i u_i'+g\sum_{j=1}^{i} d_i'+g\sum_{j=i+1}^{n}\frac{\rho_j}{\rho_i}d_j'+\frac{p_s'}{\rho_i}\right]=0, \tag{3.2}$$

for flow over a level surface. For motion with $u_i' = u_i'(x-ct)$, etc. propagating with speed c, (3.2) may be integrated to give

$$(U_i-c)\,u_i'+g\sum_{j=1}^{i} d_j'+g\sum_{j=i+1}^{n}\frac{\rho_j}{\rho_{i+1}}d_j'+\frac{p_s'}{\rho_i}=0, \tag{3.3}$$

the constant of integration on the right-hand side being taken as zero for linear waves. In the frame of reference moving with the wave, the flow is steady. In this frame the velocity of the ith layer is $U_i+u_i'-c$, and continuity gives

$$(U_i+u_i'-c)\,(D_i+d_i') = \text{constant} = (U_i-c)\,D_i. \tag{3.4}$$

Linearizing then gives

$$u_i' = -(U_i-c)\,\frac{d_i'}{D_i} \quad (i = 1,\ldots,n). \tag{3.5}$$

Eliminating u_i' between (3.3), (3.5) then gives

$$-\rho_i(U_i-c)^2\frac{d_i'}{gD_i}+\rho_i\sum_{j=1}^{i} d_j'+\sum_{j=i+1}^{n}\rho_j d_j'+\frac{p_s'}{g}=0 \quad (i = 1,\ldots,n). \tag{3.6}$$

If the upper boundary is free, we have

$$p_s' = -\rho_{n+1}g\sum_{j=1}^{n} d_j' = -\frac{\rho_{n+1}(U_n-c)^2}{1-\dfrac{\rho_{n+1}}{\rho_n}}\frac{d_n'}{D_n}. \tag{3.7}$$

Eliminating p_s' and taking the determinant of the coefficients of the d_i' yields the equation for c

$$\begin{vmatrix} \rho_1F_1^2+\rho_2F_2^2-\Delta_1\rho & \rho_2F_2^2 & \cdot\cdot & 0 & 0 \\ \rho_2F_2^2 & \rho_2F_2^2+\rho_3F_3^2-\Delta_2\rho & \cdot\cdot & 0 & 0 \\ 0 & \rho_3F_3^2 & \ddots & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \rho_{n-1}F_{n-1}^2 & 0 \\ 0 & 0 & \cdot\cdot & \rho_{n-1}F_{n-1}^2+\rho_nF_n^2-\Delta_{n-1}\rho & \rho_nF_n^2 \\ 0 & 0 & \cdot\cdot & \rho_nF_n^2 & \rho_nF_n^2-\Delta_n\rho \end{vmatrix} = 0, \tag{3.8}$$

where

$$\Delta_i\rho = \rho_i-\rho_{i+1}, \quad F_i^2 = \frac{(U_i-c)^2}{gD_i}. \tag{3.9}$$

Equation (3.8) is identical with Benton's equation (22), except that he took $\rho_{i+1} = 0$. It has $2n$ roots for c. If the upper boundary is rigid we have

$$\sum_{i=1}^{n} d_n' = 0, \tag{3.10}$$

so that

$$p_s' = \rho_n(U_n-c)^2\frac{d_n'}{D_n}. \tag{3.11}$$

Eliminating p_s' as before yields a tri-diagonal determinantal equation for c which is identical with (3.8), but with the nth row and nth column deleted (Benton's equation (45)). This equation has $2(n-1)$ roots for c.

For the purposes of this paper we will assume that all the roots for c are real, so that they represent neutrally stable internal gravity waves, unless it is stated otherwise. This is equivalent to the assumption that the flow is always stable to shear instabilities at long wavelength. We also note that there is no restriction on the values c may take. Inspection of two-layer flow, for example, shows that some wave speeds may lie between the speed of the fluid layers or perhaps be equal to them, without singularities arising. In these layered flows, critical-layer phenomena, as found in continuously stratified flows, do not exist. This implies that layered models cannot represent continuously stratified phenomena associated with critical layers of infinitesimal thickness.

For each value of c, the corresponding eigenfunctions (u_i', d_i') may be obtained from (3.5)–(3.7) for the free upper boundary, and (3.5), (3.6) and (3.10)–(3.11) for the rigid upper boundary. In both cases the matrix of coefficients for the d_i' may be written in triangular form, so that computation is simple. We may note that although the eigenvalues may be thought of as occurring in pairs (two for each mode), the structure of the members of the corresponding pair of eigenfunctions will differ from each other unless all the U_i are equal.

We now show the relationship between these waves and the conditions at a topographic extremum in hydrostatic flow. Differentiating equations (2.8), (2.9) for steady flow we obtain

$$\left.\begin{aligned} d_i\frac{\mathrm{d}u_i}{\mathrm{d}x}+u_i\frac{\mathrm{d}d_i}{\mathrm{d}x} &= 0\\ u_i\frac{\mathrm{d}u_i}{\mathrm{d}x}+g\sum_{j=0}^{n}\rho_{ij}\frac{\mathrm{d}d_j}{\mathrm{d}x}+\frac{1}{\rho_i}\frac{\mathrm{d}p_s}{\mathrm{d}x} &= 0\end{aligned}\right\}\quad (i=1,\ldots,n). \tag{3.12}$$

Eliminating $\mathrm{d}u_i/\mathrm{d}x$ then gives

$$-\rho_i\frac{u_i^2}{gd_i}+\rho_i\sum_{j=1}^{i}\frac{\mathrm{d}d_j}{\mathrm{d}x}+\sum_{j=i+1}^{n}\rho_j\frac{\mathrm{d}d_j}{\mathrm{d}x}+\frac{1}{g}\frac{\mathrm{d}p_s}{\mathrm{d}x}=-\rho_i\frac{\mathrm{d}h}{\mathrm{d}x}\quad (i=1,\ldots,n). \tag{3.13}$$

For a free upper boundary we also have

$$\frac{\mathrm{d}p_s}{\mathrm{d}x}=-g\rho_{n+1}\left(\sum_{j=1}^{n}\frac{\mathrm{d}d_j}{\mathrm{d}x}+\frac{\mathrm{d}h}{\mathrm{d}x}\right), \tag{3.14}$$

so that

$$\frac{\mathrm{d}p_s}{\mathrm{d}x}=\frac{-\rho_{n+1}u_n^2}{\left(1-\dfrac{\rho_{n+1}}{\rho_n}\right)d_n}\frac{\mathrm{d}d_n}{\mathrm{d}x}. \tag{3.15}$$

Hence (3.13) may be written

$$\begin{bmatrix} \rho_1(1-f_1^2) & \rho_2 & \cdot\cdot & \rho_n-\dfrac{\rho_{n+1}f_n^2}{\left(1-\dfrac{\rho_{n+1}}{\rho_n}\right)}\\ p_2 & \rho_2(1-f_2^2) & & \\ \rho_3 & \rho_3 & \ddots & \\ \vdots & \vdots & & \\ \rho_n & \rho_n & \cdot\cdot & \rho_n-\left(\rho_n+\dfrac{\rho_{n+1}}{1-\dfrac{\rho_{n+1}}{\rho_n}}\right)f_n^2 \end{bmatrix}\begin{bmatrix}\dfrac{\mathrm{d}d_1}{\mathrm{d}x}\\ \\ \vdots\\ \\ \dfrac{\mathrm{d}d_n}{\mathrm{d}x}\end{bmatrix}=-\frac{\mathrm{d}h}{\mathrm{d}x}\begin{bmatrix}\rho_1\\ \\ \vdots\\ \\ \rho_n\end{bmatrix} \tag{3.16}$$

where
$$f_i^2 = \frac{u_i^2}{gd_i}. \tag{3.17}$$

If we identify f_i^2 with F_i^2 for all i, the left-hand side of (3.16) for $\mathrm{d}d_i/\mathrm{d}x$ is identical with that for (3.6), (3.7) for d_i'. It follows that when
$$\frac{\mathrm{d}h}{\mathrm{d}x} = 0, \tag{3.18}$$
we must have either
$$\frac{\mathrm{d}d_i}{\mathrm{d}x} = 0, \quad \text{all } i, \tag{3.19}$$
or the determinant of coefficients in (3.16) must vanish. If we write U_i, D_i for u_i, d_i, the latter implies that we must have a solution to (3.8) with
$$c = 0. \tag{3.20}$$
In other words, the speed of one of the internal wave modes must be zero. This is the usual definition of the term 'critical flow', and we will employ this definition here. There are $2n$ different possible critical flow states.

If we differentiate (3.16) we may show that at a topographic inflexion point where $\mathrm{d}h/\mathrm{d}x$, $\mathrm{d}^2h/\mathrm{d}x$ are both zero, if $\mathrm{d}d_i/\mathrm{d}x$ vanishes there for all i, $\mathrm{d}^2d_i/\mathrm{d}x^2$ must also vanish, so that all the interfaces also have inflexion points there. On the other hand, if $\mathrm{d}h/\mathrm{d}x$ and $\mathrm{d}d_i/\mathrm{d}x$ (all i) vanish but $\mathrm{d}^2h/\mathrm{d}x^2$ does not, $\mathrm{d}^2d_i/\mathrm{d}x^2$ (all i) will also be non-zero, so that a topographic extremum implies symmetric streamlines over the topography, for non-critical flow.

For a rigid upper surface we will have
$$\sum_{i=1}^{n} \frac{\mathrm{d}d_i}{\mathrm{d}x} + \frac{\mathrm{d}h}{\mathrm{d}x} = 0, \quad \frac{\mathrm{d}p_s}{\mathrm{d}x} = \frac{\rho_n u_n^2}{d_n}\frac{\mathrm{d}d_n}{\mathrm{d}x}, \tag{3.21}$$
in place of (3.14), (3.15). From the above argument, *mutatis mutandis*, we obtain the same results, namely that when $\mathrm{d}h/\mathrm{d}x = 0$ we must have either
$$\left.\begin{aligned} \frac{\mathrm{d}d_i}{\mathrm{d}x} &= 0 \quad (i = 1, \ldots, n), \\ \text{or} \qquad c &= 0, \end{aligned}\right\} \tag{3.22}$$
for some internal wave mode.

The condition of critical flow at an obstacle crest imposes a restriction on the flow, which may be expressed as a single algebraic equation relating the variables of all the layers. For a single layer this is a very strong restriction, but this strength decreases as the number of layers (or 'degrees of freedom') increases.

In hydraulics it is common usage to term a flow 'supercritical' or 'subcritical'. For uni-directional stratified flows we may say that the flow is sub(super)critical with respect to a particular (ith) wave mode if that mode may (may not) propagate against the stream, i.e. $c_i < 0$ ($c_i > 0$). The term may be applied to any particular point in the flow, or to the flow as a whole. Since increasing the obstacle height generally increases the values of c_i for all i, if the flow is subcritical at the obstacle crest it is subcritical everywhere, for that mode. Of course, a flow may be subcritical with respect to some modes and supercritical with respect to others.

A special type of linear disturbance which is very important in the present discussion is the 'columnar disturbance mode'. In the hydrostatic long-wave model employed here, these modes consist of a small change in the velocity and density profiles which propagates at the linear wave speed without changing shape. The changes in velocity and density have the structure of the appropriate wave mode. Propagation of the columnar disturbance mode past a certain point therefore results in a permanent change to the mean flow state. If the long-wave approximation is not made, the propagation change has the form of an evolving dispersive wave (McEwan & Baines 1974), but the overall effect is the same.

4. Hydraulic jumps

For our general model of stratified flow over topography, treated as an initial-value problem, we need to consider the possible existence of hydraulic jumps and equations which relate the conditions across them. Hydraulic jumps are nonlinear structures which propagate without changing, and although they have only been satisfactorily observed and described for one- and two-layer flows, it is natural to assume that they exist in multi-layer flows also. In our hydrostatic flow model they may be modelled as discontinuities between one uniform stream and another (see figure 3*a*), and they usually involve a dissipation of energy. Consequently, the energy conservation equation (2.9) is not applicable in general. Numerical procedures for computing hydraulic jump properties in hydrostatic layered flows have been given by Su (1976), and we follow his formalism here.

Hydraulic jumps may be turbulent or laminar and involve non-hydrostatic features. For these reasons it is necessary to make three assumptions (or restrictions) in order to accommodate them in our hydrostatic model of layered flow. These assumptions are numbered (i)–(iii) below.

We assume: (i) each layer maintains its identity, density, and mass flux through the jump. This implies that mixing in the jumps is negligible; this will be so if the layers are immiscible or if the jumps are sufficiently weak. With this assumption ρ_i is constant through the jump, and if U_i, D_i denote velocity and layer thickness upstream and u_i, d_i the same downstream respectively for the ith layer, in a frame of reference in which the jump is stationary, we have

$$U_i D_i = u_i d_i. \tag{4.1}$$

(ii) The flow in the jump is hydrostatic, or at least sufficiently so for our purposes. The steady-state momentum equation obtained from (2.1), (2.4) for the ith layer is then

$$\frac{\partial}{\partial x}(\rho_i d_i u_i^2 + \rho_i g \tfrac{1}{2} d_i^2) + g\rho_i \sum_{\substack{j=1 \\ j\neq i}}^{n} \rho_{ij} d_i \frac{\partial d_j}{\partial x} + d_i \frac{\partial p_s}{\partial x} = 0. \tag{4.2}$$

Integrating this equation across the jump yields

$$d_i u_i^2 - D_i U_i^2 + \tfrac{1}{2} g(d_i^2 - D_i^2) + \int_{\text{upstream}}^{\text{downstream}} \left(g \sum_{\substack{j=1 \\ j\neq i}}^{n} \rho_i d_i \, \mathrm{d}d_j \right) + \left(\frac{d_i}{\rho_i} \mathrm{d}p_s \right) = 0, \tag{4.3}$$

which may be written

$$d_i u_i^2 - D_i U_i^2 + \tfrac{1}{2} g(d_i^2 - D_i^2) + g \sum_{\substack{j=1 \\ j\neq i}}^{n} \rho_{ij} \overline{d_i}(d_j - D_j) + \frac{\overline{d_i}}{\rho_i}(p_s - P_s) = 0, \tag{4.4}$$

where $\overline{d_i}$ denotes the mean value of d_i in the jump.

(iii) The mean value of the ith-layer thickness in the jump is given by

$$\overline{d_i} = \tfrac{1}{2}(D_i + d_i). \tag{4.5}$$

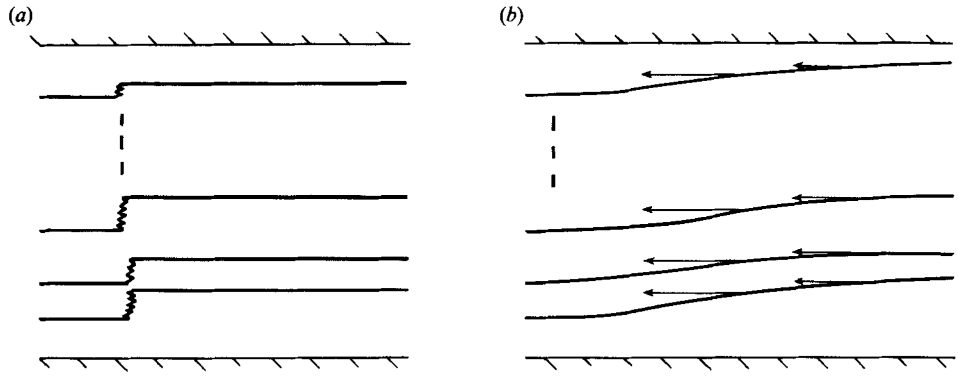

FIGURE 3. (*a*) Representation of a hydraulic jump for the lowest mode. (*b*) Representation of a rarefaction for the lowest mode. The arrows indicate that the leading part of the disturbance travels faster than the trailing part.

Combining (4.1), (4.4), and (4.5) and eliminating u_i, we obtain Su's jump equations

$$\frac{U_i^2}{gD_i}\frac{2\xi_i}{(1+\xi_i)(2+\xi_i)} = \sum_{j=1}^{n} \rho_{ij}\frac{D_j}{D_i}\xi_j + \frac{1}{g\rho_i D_i}(p_s - P_s) \quad (i = 1, \ldots, n), \tag{4.6}$$

where

$$\xi_i = \frac{d_i}{D_i} - 1.$$

The above three assumptions are quite plausible if the jumps are weak, and are equivalent to those proposed by Yih & Guha (1955) and employed by Houghton & Issacson (1970), Su (1976), and the author (Baines 1984). Recently, Chu & Baddour (1977) and Wood & Simpson (1984) have put forward different assumptions for two-layered flows which may well be superior. However, it is more difficult to apply their scheme to jumps in multi-layered flows because it requires a knowledge of which layers are contracting. Also, differences in results for weak jumps from the two schemes are small, and observations have not been able to distinguish between them. Further study is required, and pending the outcome, use of the first scheme is proposed.

Simple algorithms for computing jumps based on (4.6) have been given by Su (1976, §7) for both rigid and free upper surfaces. He also presented some numerical examples. These algorithms are summarized in Appendix A. Where jumps exist, as their amplitudes approach zero their properties (speed and structure) approach those of linear waves (or more exactly, columnar disturbance modes, as described in the previous section). In the applications described below, this property enables the non-uniqueness of the jump equations (as described by Yih & Guha and Su) to be resolved. However, there may or may not be a possible jump associated with a particular linear wave mode; the situation for two-layer flows is illuminating, and is discussed in detail in Baines (1984).

5. The general method

We now describe a flexible procedure for the description and computation of the flow that results from the introduction of an obstacle into a known stable stratified shear flow, bearing in mind the approximation and assumptions of §1. The procedure is a generalization of results from two-layer studies and has been applied to a number of other situations. It is embodied in a set of computer programs written in FORTRAN. These are available from the author, but they may be constructed by following the steps described below.

The upper surface may be rigid or free; we shall assume that it is rigid for definiteness. The free surface case then follows, *mutatis mutandis*. We may define a mean velocity $\bar{U}$ by

$$\bar{U} = \frac{1}{D}\int_0^D U(z)\,\mathrm{d}z = \frac{1}{D}\sum_{i=1}^{n} U_i D_i, \tag{5.1}$$

which is taken to be positive, and an overall initial Froude number F_0 by

$$F_0 = \frac{\bar{U}}{\bar{U}-c_1},$$

where c_i is the velocity of the fastest linear wave mode propagating against the stream in the frame of the obstacle (the rest frame) in the undisturbed flow. c_1 may be positive or negative; linear waves can only propagate upstream when it is negative, so that $0 < F_0 < 1$. In these circumstances the flow is said to be 'subcritical' with respect to mode 1. If instead c_1 is positive so that $F_0 > 1$, the flow is said to be 'supercritical' with respect to all modes. In general these wave speeds will be functions of horizontal position because of changes in the basic flow, and over the obstacle (and in particular at the crest) their values will be different from the upstream values.

In some circumstances the procedure requires an additional assumption (termed assumption A), namely that the presence of the topography does not result in flows on the upstream side which move upstream relative to the obstacle (i.e. have negative velocity) in the steady state, at any level. This is based on a large number of laboratory observations by the author, and is equivalent to saying that no fluid moves from the downstream side of the obstacle to the upstream side. It is also generally observed that, for any given steady-state flow, if the obstacle height is increased slightly the resulting change to the flow field is such that the velocity of the upstream fluid approaching the obstacle at the lowest moving level is decreased, provided that the above assumption is not violated in the process.

To obtain the flow over an obstacle of given height, one begins with an obstacle of small or zero height where the steady-state flow is known. The procedure then consists of calculating the successive changes to the flow caused by incremental increases in the height of the obstacle, up to the desired value. We describe the procedure in three stages, each corresponding to progressively higher obstacles.

Stage 1

We assume initially that at least one mode may propagate upstream, so that $0 < F_0 < 1$, and we imagine that an obstacle with small height is introduced into this flow. If the obstacle height h_m is less than a critical height h_c the new steady-state flow will be the same as the undisturbed one except over the obstacle. This flow state is shown schematically in figure 4(*a*). These steady-state changes to the flow and the value of h_c may be calculated from (2.8), (2.9), and algorithms for doing this are given in Appendix B. From the discussion of §3 we have either

$$\left.\begin{aligned} \frac{\mathrm{d}d_j}{\mathrm{d}x} &= 0 \quad \text{all } j \\ \text{or} \qquad c_i &= 0 \quad \text{for some } i \end{aligned}\right\} \quad \text{at } \frac{\mathrm{d}h}{\mathrm{d}x} = 0,$$

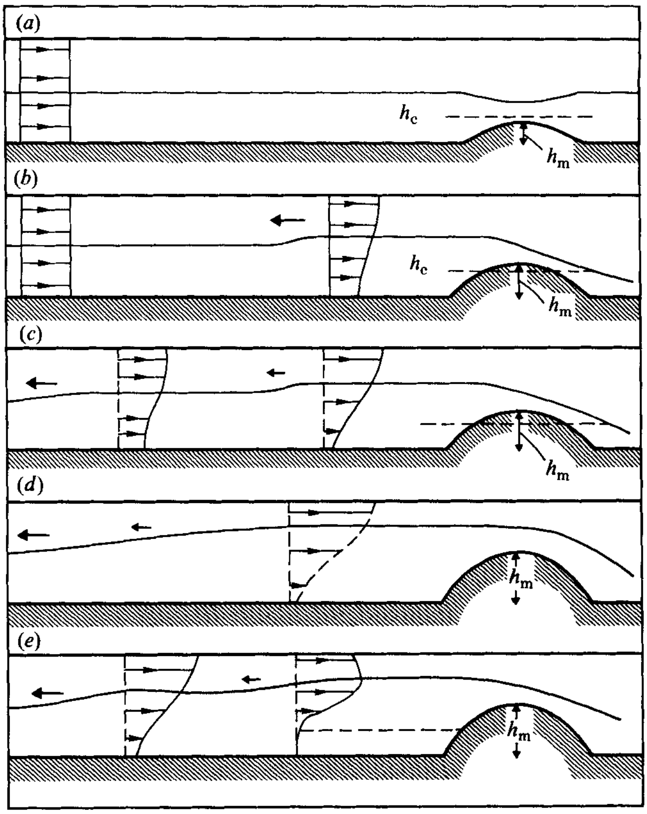

FIGURE 4. Schematic diagrams of flow states illustrating the method of flow calculation. An obstacle is introduced into a flow with an initially uniform velocity profile (but an unspecified density profile) with $F_0 < 1$. Upstream velocity profiles and a typical streamline are shown. (*a*) $h_m < h_c$, the critical height for this value of F_0; the upstream and downstream steady state is unchanged by the topography. (*b*) h_m has been increased to slightly above h_c; flow at the obstacle crest adjusts to a new critical state and causes a columnar disturbance mode to be sent upstream, altering the upstream flow state. (*c*) a further small increase in h_m causes the process to be repeated; the second disturbance may travel faster (resulting in a jump) or slower (resulting in a rarefaction) than the previous one. (*d*) If h_m is sufficiently large the flow may become critical just upstream and supercritical over the obstacle, as shown here, or (*e*) a fluid layer may become blocked. This usually requires more than one upstream mode, as shown here. As $h_m$increased from zero to its present value the flow passed through states *a*, *b*, *c* and *d* for the first mode, and a second mode has now become critical at the obstacle crest.

and that for $h_m = h_c$, $c_i = 0$ at the obstacle crest for some i. Hence for $h_m < h_c$, $\mathrm{d}d_j/\mathrm{d}x = 0$ at $\mathrm{d}h/\mathrm{d}x = 0$ for all j. It is not possible for higher obstacles (i.e. $h_m > h_c$) to be present with this undisturbed upstream flow in steady-state conditions, and hence, if h_m is increased above h_c, changes must take place.

Stage 2

If the obstacle height is increased very slightly (infinitesimally) above h_c by an amount Δh, the flow will adjust locally so that is again critical at the obstacle crest for the same ith mode. This will cause a small linear disturbance, in the form of a columnar disturbance mode, to be sent upstream which will alter slightly the oncoming velocity and density profiles in the new steady state. This disturbance will have the structure of the same ith mode, will have amplitude Δa (say), and will travel upstream at the long-wave speed of the the ith mode, c_i. This is shown schematically in figure 4(*b*).

If the obstacle height is then increased further by an infinitesimal amount, this process will be repeated: the flow will adjust to a slightly different state at the obstacle crest, which will again satisfy $c_i = 0$ there, and a linear columnar disturbance mode will propagate upstream at the new linear wave speed, altering the oncoming flow which approaches the obstacle. Here, however, we must make an important distinction between two different cases. The propagation speed of the new upstream disturbance may be written $c_i + \Delta c_i$, where Δc_i denotes the difference in speed from the previous value. This speed will be slightly different because the second disturbance will propagate on the slightly modified flow behind the first disturbance. Δc_i may be positive or negative. If Δc_i is positive or zero, the second disturbance will never catch up to the previous one, and the new upstream flow and the flow over the obstacle are already determined (figure 4*c*). If, however, $\Delta c_i < 0$, the second disturbance will travel faster upstream and will catch up with the first one and increase its amplitude. In effect, this will form an infinitesimal hydraulic jump. As discussed in §4, a hydraulic jump travels at a speed which is dependent on its amplitude, and jump conditions may be found which determine the structure of the flow on its downstream side in terms of the upstream conditions and the jump amplitude. Once the jump has formed, this structure will in general be slightly different from that which was present behind the second upstream disturbance. This difference will then be communicated back to the flow in the vicinity of the obstacle and cause further adjustments there. These changes will in turn affect the jump, and the flow will finally reach a steady state when the jump amplitude is adjusted so as to be consistent with a critical flow state at the obstacle crest.

If the obstacle height is increased still further and successive values of Δc_i all have the same sign, these processes will be repeated. The result in the first case ($\Delta c_i > 0$) will be a succession of upstream disturbances which become increasingly spread out, forming a rarefaction (see figure 3*b*), and the result in the second case ($\Delta c_i < 0$) will be a progressively larger hydraulic jump. These two different types of upstream motion and the conditions determining them may be summarized by saying that

$$\frac{\mathrm{d}c_i}{\mathrm{d}a} < 0 \quad \text{implies a hydraulic jump,}$$

and

$$\frac{\mathrm{d}c_i}{\mathrm{d}a} > 0 \quad \text{implies a rarefaction,}$$

where c_i denotes the upstream propagation speed of the ith mode, which is the mode that is critical at $\mathrm{d}h/\mathrm{d}x = 0$, and a denotes the upstream amplitude of this mode. Expressions for $\mathrm{d}c_i/\mathrm{d}a$ may be obtained in terms of the mean flow properties and the structure of the relevant eigenfunction, and these are derived in Appendix C. It is important to note that c_i and a here are cumulative variables, in the sense that the following disturbances propagate on and add to previous ones. The structure of the corresponding eigenfunction also changes continuously. If the upstream amplitudes of these disturbances are small the resulting flows calculated assuming one or the other flow type will be smilar, but as the amplitude increases the flow properties will diverge.

Both of these physical processes may be calculated numerically. The procedure of Lee & Su (1977) given in Appendix B may be applied for obstacles up to the critical height (Stage 1). For higher obstacles the procedure is as follows. For the initial incident upstream flow, we first obtain the eigenvalues (giving the wave speeds) and the corresponding eigenfunctions (giving the structure) for the internal wave modes which may propagate upstream. (Numerical procedures for obtaining such eigenvalues and eigenfunctions are well known.) The mode with the smallest negative (i.e. upstream) speed will be the mode which is critical at the obstacle crest, because c_i increases over the obstacle. It is this mode which will be propagated upstream as a columnar disturbance mode if the obstacle is slightly higher. We therefore add a sufficiently small increment of this mode, in the form of the horizontal velocity profile and associated density change, to the upstream flow, and then apply the procedure of Appendix B to obtain the new critical obstacle height. If this critical height is greater than the previous one, the correct sign of the upstream disturbance has been chosen. If it is less, the sign of the disturbance should be reversed and the new critical height found. One then solves for the new eigenvalues. If the eigenvalue for the upstream wave speed for the same mode as before is more negative than before, one must use the procedure appropriate for jumps (see next paragraph). If it is not, then we proceed by simply repeating the previous steps, adding an increment of the associated mode to the upstream profile and solving for the corresponding critical height. This process may then be repeated again and again and is justifiable provided the new upstream wave speed at each step is more positive than or equal to the previous one.

If, at the second calculation, the upstream wave speed for the relevant ith mode is less (i.e. more negative) than the first calculated value, a jump procedure is appropriate. This involves assuming the presence of a jump, initially at a very small amplitude and with a speed approximately equal to c_i. Conditions for such jumps and their properties may be calculated using the procedure of Su outlined in Appendix A. With an assumed upstream jump of given amplitude, the horizontal velocity and density profiles downstream of the jump and upstream of the obstacle will be determined. We then utilize the procedure of Appendix B to find the new critical height of the obstacle in this new oncoming stream, and, all things being correct, it will be higher than the previous critical height. Calculations of this nature with a single jump may be used to determine the flow over obstacles with a range of heights greater than h_c. It is not correct to hypothesize a succession of small jumps, because these will accumulate into a single jump in the steady state, and the two downstream states will be different. The calculations for hydraulic jumps involve a requirement that energy be dissipated in the jump, and this necessary condition is given in Appendix B. For the two-layer system described in Baines (1984), this energy loss becomes zero when the jump reaches its maximum amplitude.

If $\mathrm{d}c_i/\mathrm{d}a < 0$ initially, as the jump amplitude is increased the value of this derivative downstream of the jump may decrease to zero and become positive. When this occurs it is necessary to change from the 'jump procedure' to the 'rarefaction procedure' at the point where the derivative becomes zero. The upstream disturbance will therefore consist of a hydraulic jump followed by a rarefaction. The reverse case, in which $\mathrm{d}c_i/\mathrm{d}a > 0$ initially, giving rarefactions, and then changes sign, is conceptually possible but seems uncommon. It requires c_i to have a maximum value as the amplitude of the disturbance increases, and it does not occur in any of the systems studied to date. If it should occur, rarefactions would be calculated to the point at which $\mathrm{d}c_i/\mathrm{d}a$ became negative; as c_i progressively decreased, it would then be necessary to calculate the jump by returning to and starting from the point in the rarefaction where c_i had the same value as the new jump speed, so that the growing

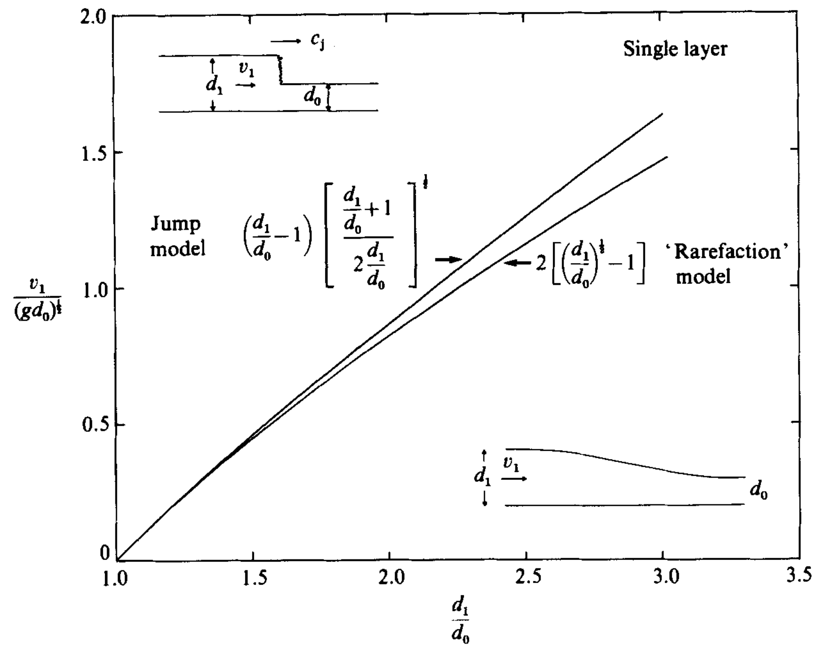

FIGURE 5. Downstream fluid velocity v_1 as a function of disturbance height d_1 in a single layer, calculated for a jump (top left) and a rarefaction (bottom right) moving into fluid at rest.

hydraulic jump would progressively erode the rarefaction. Since one begins by choosing a jump amplitude and then calculates a speed, an iterative scheme would probably be necessary.

In some situations dc/da may decrease and then increase (or vice versa) over a small range. Strictly, this requires employing a jump and then a rarefaction, but for computational reasons this may be inefficient or convenient. For the purpose of determining the steady-state flow is is often more convenient in general to use a rarefaction model in place of a jump model, as the former has been found to be easier to treat numerically. This is justified provided that the error involved is small. An indication of the magnitude of such an error can be obtained by examining the prototype system of a single layer. Referring to figure 5, for a jump moving into fluid at rest with speed C_J the equations are

$$\frac{C_J}{(gd_0)^{\frac{1}{2}}} = \left(\frac{d_1/d_0(1+d_1/d_0)}{2}\right)^{\frac{1}{2}}, \tag{5.2}$$

$$(C_J - v_1)\, d_1 = C_J\, d_0, \tag{5.3}$$

so that the fluid velocity v_i behind the jump is related to the jump height by

$$\frac{v_1}{(gd_0)^{\frac{1}{2}}} = \left(\frac{d_1}{d_0} - 1\right)\left(\frac{1+d_1/d_0}{2d_1/d_0}\right)^{\frac{1}{2}}. \tag{5.4}$$

Note that here the jump and the fluid behind it are both moving in the positive x-direction, in contrast to the other situations in this paper. For a rarefaction on the other hand, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(v_1 \pm 2(gd_1)^{\frac{1}{2}}) = 0 \quad \text{on} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = v_1 \pm (gd_1)^{\frac{1}{2}}, \tag{5.5}$$

so that $$v_1 - 2(gd_1)^{\frac{1}{2}} = \text{constant} = -2(gd_0)^{\frac{1}{2}}, \tag{5.6}$$
on a simple wave expanding into fluid at rest of depth d_0 (we here ignore the fact that if $d_1 > d_0$, this wave will steepen into a jump). Hence the fluid velocity behind the 'rarefaction' is given by

$$\frac{v_1}{(gd_0)^{\frac{1}{2}}} = 2\left[\left(\frac{d_1}{d_0}\right)^{\frac{1}{2}} - 1\right]. \tag{5.7}$$

Equations (5.4) and (5.7) are both plotted in figure 5, which shows that they agree quite well even to quite large amplitudes. Hence the character of the disturbance only affects the flow state behind it when the disturbance amplitude is very large, so that the rarefaction procedure is equivalent to the jump procedure at moderate amplitudes. Similar agreement is expected in more complex layered systems.

Stage 3

The above procedures may be followed to obtain the flow over progressively higher obstacles until one of two things happens. These are (i) the flow immediately upstream of the obstacle may become critical (with respect to the ith mode, so that $c_i = 0$ just upstream) or (ii) the velocity U_i of some fluid layer (usually, but not always, that of the lowest layer, U_1) may become zero just upstream. We now discuss each of these situations in turn.

(i) Critical flow upstream, c_i increases to zero. When c_i becomes zero upstream the flow over the topography must be supercritical with respect to this mode. The flow over yet higher obstacles may now be calculated using the procedure of Appendix B with the flow upstream fixed, giving symmetric supercritical flow (in the ith mode) at the obstacle crest. If the ith mode is in fact the fastest upstream mode (i.e. $i = 1$), this will be applicable for obstacle heights up to the maximum $h_m = D$. The flow is shown schematically in figure 4(d). If the ith mode is not the fastest, then as the obstacle height is increased the slowest mode still propagating upstream (the $(i-1)$th) will become critical at the obstacle crest (i.e. $c_{i-1} = 0$ there) at some value for the obstacle height. This constitutes a new critical height, but now it is for the $(i-1)$th mode. To obtain in the flow for higher obstacles it is now necessary to alter the upstream flow profile, and this is done by repeating the procedure described in Stage 2, but now involving the $(i-1)$th mode rather than the ith. An example where this occurs is given in figure 7.

(ii) Blocking. Assumption A above implies that when the velocity of a fluid layer is reduced to zero, changes must take place in the character of the upstream disturbances. If the blocked layer is the lowest layer (as is most common), further increases in the obstacle height must result in upstream disturbances which (after their passing) keep layer 1 at rest whilst (probably) reducing the velocity of layer 2. For this to occur, the upstream disturbances must be more complicated than previously. In general they will consist of two disturbances – a faster mode and a slower mode, of which either may be a jump or a rarefaction, and which together result in the lowest layer remaining at rest, but with an altered thickness. The use of the procedures described above with such double upstream disturbances was found to be difficult and tedious. For this reason, blocked flows were not investigated in any detail. However, some practical approximations may be noted. For the upstream disturbances, for all cases where $\Delta_1 \rho / \rho_1 \ll 1$ it may be easily demonstrated that inertial effects in the bottom (nearly) stagnant upstream layer are negligible. The layer may then be termed 'inert' on 'isostatic'; its thickness adjusts to changing pressure above to keep the pressure in the bottom layer constant. This also implies that the upper surface of the bottom layer acts as a lower free surface to the moving layers above. This provides an approximation which may be incorporated into the equation of §3 for the upstream wave speeds and corresponding eigenfunctions, by simply omitting the terms containing F_1 as a factor; the upstream disturbances are reduced from two modes to a single mode with a free lower surface. When, for higher obstacles, the second layer is also brought to rest, the isostatic approximation for the bottom two layers requires that terms containing F_1 and F_2 both be omitted. Once the second layer becomes blocked it 'shields' the bottom layer from changes occurring in the moving layers above, so that only the uppermost stationary layer is affected by them. This procedure may be extended to any number of blocked layers. However, although the isostatic approximation for a blocked layer is a useful simplification, upstream disturbances with a free lower boundary do not necessarily conserve the mass flux in the channel. Therefore, for large amplitude disturbances with a blocked layer further modifications to the procedure are required, and these are discussed for the special case of the three-layer system in the next section.

In the discussion so far it has been assumed that $F_0 < 1$. If $F_0 > 1$ the flow will be supercritical with respect to all modes for sufficiently small obstacles, and it may be calculated using (2.8), (2.9) and Appendix B. If F_0 is large enough, the flow may remain supercritical over all obstacles as the height is increased to the maximum possible (the total depth D in finite depth systems). Alternatively, as h_m is increased the flow may become critical at the obstacle crest for the fastest mode, giving a critical value $h_m = h_c$. For higher obstacles, disturbances must be sent upstream to alter the velocity and density profiles. Since $F_0 > 1$, however, these cannot take the form of linear waves travelling at speed c_1. The results from two-layer studies indicate that the upstream disturbance must take the form of a jump of finite amplitude, large enough to propagate against the oncoming stream. This jump may be calculated by the same procedures as given above for $F_0 < 1$, with critical flow at $h = h_m$. Downstream of the jump the flow will be subcritical with respect to the obstacle for the fastest mode, so that the procedures described above for $F_0 < 1$ will apply for larger obstacles here also.

The fact that a jump appears at finite amplitude suggests that, for smaller obstacles, it may also exist at smaller amplitudes. This is in fact the case for two-layer systems, implying a hysteresis phenomenon where two steady flow states (supercritical flow, or flow with an upstream jump) may exist for the same external flow parameters, and the state which is actually obtained depends on the history of the flow. This is discussed in detail in Baines (1984), and we should look for these phenomena in any system where jumps are possible.

For systems with $F_0 < 1$ (and possibly even $F_0 > 1$), this type of two-state behaviour may well exist for higher order (i.e. slower) modes than the fastest, when jumps associated with these modes are possible. Given the wide range of possible stable stratified shear flows, all types of combinations of phenomena mentioned above are conceivable, and a thorough analysis of the nonlinear dispersion properties of a system will be necessary to describe its flow properties.

6. Two examples

To illustrate the above general procedure and the nature of the results obtained, we here briefly describe the results from two examples. For both of these, the overall density variation is assumed to be small (i.e. $\Delta\rho/\bar{\rho} \ll 1$). The examples are as follows.

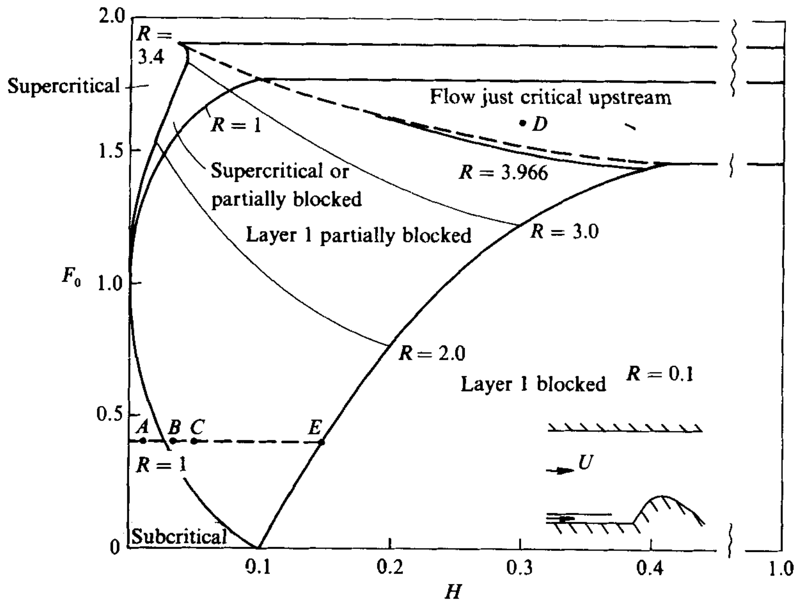

FIGURE 6. Steady-state flow properties of a two-layer system with a rigid upper boundary in terms of the initial Froude number F_0 and $H = h_m/D$, where D is the total depth. Here $\Delta\rho/\bar{\rho} \ll 1$, $r = d_{10}/D = 0.1$ where d_{10} is the undisturbed depth of the lower layer, and $R = d_{11}/d_{10}$ is the relative increase in depth of the lower layer due to upstream disturbances caused by the introduction of the obstacle. Points denoted A, B, C, D and E correspond, loosely, to flows a, b, c, d and e of figure 4. 'Partially blocked flow' refers to flow with a reduced mass flux in the lower layer. The dashed line denotes the boundary of the region where the flow is critical upstream. A hysteresis region exists when $F_0 > 1$, where the flow may be supercritical or partially blocked, as discussed in Baines (1984) where more details are given for two-layer systems.

(i) A two-layer system with a rigid upper boundary, uniform velocity profile, and the lower thickness equal to 10% of the total depth. Figure 6 shows the regions of (F_0, H)-space where the various different types of upstream flow are obtained using the procedure of §5. This diagram is similar to figure 16 of Baines (1984) but is more complete, covering obstacles up to the maximum height. If we take $F_0 = 0.4$ (for example) and increase the obstacle height from zero, the flow is subcritical until h_m reaches the critical height at the solid curve. Up to this point the flow resembles state (a) of figure 4. If the height is increased above the critical value to point B a small amplitude jump is sent upstream, forced by the requirement for a critical condition at the obstacle crest. A larger jump is obtained at point C, and so on to point E, where the lower layer is totally blocked. Further increases in the obstacle height do not change the upstream flow. The total (F_0, H)-diagram may be constructed in this way; upstream jumps are obtained up to the maximum height ($R = 3.965$) or to the dashed line (where the upstream flow is critical). Rarefactions are produced to the right of the curve $R = 3.965$, up to the dashed line. For $F_0 > 1$ there is a hysteresis region where the flow may be supercritical or partially blocked. The procedure described above will only give the 'supercritical' solution in this region, and to obtain the other solution it is necessary to initiate the flow in a different manner, as described in Baines (1984).

(ii) A three-layer system, where the velocity profile is initially uniform and the layer depths and density increments are equal. The flow properties in terms of F_0, $H = H_m/D$ calculated by this procedure are shown in figure 7.

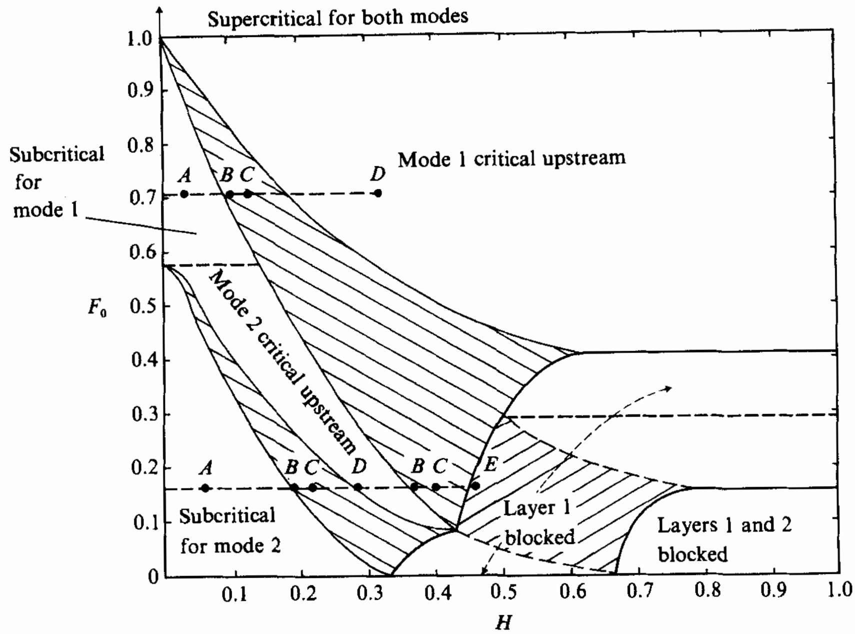

FIGURE 7. Steady-state flow properties in terms of F_0 and $H = h_m/D$ of a three-layer system with a rigid upper boundary; the layer depths, their fluid velocities and density increments are all initially equal and $\Delta\rho/\bar{\rho} \ll 1$. This system has two modes, compared with one for the system in figure 6. In the shaded regions the upstream disturbances increase with increasing obstacle height. The points A, B, C, D and E correspond to the states a, b, c, d and e of figure 4.

The shaded parts of the diagram illustrate the regions where the upstream disturbance increases with obstacle height, and the non-shaded parts of the regions where it remains constant as H increases. This system has two vertical modes rather than one (in example (i)) and this is reflected in the nature of the diagram. Points A, B, C, D and E represent flow states a, b, c, d and e of figure 4. The upstream disturbances are rarefactions if layer 1 is not blocked, and jumps (mostly) after layer is blocked. The flow is always supercritical if $F_0 > 1$.

When layer 1 was blocked the isostatic-lowest-layer approximation was used initially, but with increasing obstacle height this resulted in an increased loss of mass flux through the channel. To accommodate this, when layer 1 was blocked the following approximate procedure was adopted after rejecting several others. (i) The upstream disturbance was first calculated with the assumption that layer 1 was rigid; (ii) the change in thickness of layer 1 was then obtained from the hydrostatic relationship, with Bernoulli's equation being assumed for the uppermost layer to obtain the change in pressures at the top-most surface of the channel; (iii) from the new value of d_1, d_2 and d_3 were corrected in the same proportion, and then u_2, u_3 obtained by conservation of mass in each layer. This procedure has not been formally justified but the results obtained were plausible, satisfied all the constraints, and were generally consistent with observations of the lowest layer depth described in Baines & Guest (1988).

7. Summary and discussion

I have described a flexible procedure for calculating the flow which results from the introduction of a single long obstacle into an arbitrary stable stratified shear flow which does not permit internal wave energy to escape through an upper boundary. The procedure is based on a generalization of results obtained from the study of two-layer flow and may be summarized as follows. For any given flow, the steady-state changes caused by introducing an obstacle will be restricted to the vicinity of the obstacle if the maximum obstacle height h_m is less than a critial value h_c. When $h_m = h_c$, the flow is critical at the obstacle crest. If h_m is increased above h_c, disturbances are sent upstream which alter the oncoming velocity and density profiles so as to keep the flow critical at the obstacle crest. These upstream disturbances may be of two types – a hydraulic jump, which occurs when larger disturbances travel upstream faster than smaller ones, and a rarefaction, when the reverse is the case. If the obstacle height is increased further these disturbances will also increase (and may change from one type to the other) until either the fluid at some level becomes blocked (i.e. it cannot pass over the obstacle) or the flow immediately upstream becomes critical for the relevant internal wave mode. If the obstacle height is increased still further, these processes may be repeated for other modes. An assumption based on empirical observations has been used in determining the nature of the forced upstream disturbances, namely that no fluid moves upstream relative to the obstacle in the steady state. The solution determined by this procedure, based on incremental increases in the obstacle height, will be *the* solution if it is unique. However it is known that under some circumstances (notably when jumps may be present), more than one steady flow state may be possible, and the history of the flow will be important in determining which state occurs.

Numerical procedures have been indicated for these various possibilities. The results have been verified for two-layer systems and also (to a lesser extent) for a three-layer and a continuously stratified system (Baines & Guest 1987). One important inference from the two-layer studies is that, at least in this case, it is a justifiable approximation to model jumps as discontinuities, as described in §3, although the jumps themselves may have complicated internal structure in the form of turbulent or undular bores. The familiar balance between nonlinearity and linear wave dispersion which controls solitary waves may also control bore structure, but provided they propagate as steady-state phenomena, it is justifiable to represent then as travelling discontinuities as far as the remainder of the flow is concerned.

The use of the word 'rarefaction' deserves comment. The work is used to imply that the disturbance is being rarefied or thinned out by elongation, rather than the fluid itself. Other terms such as 'expansion wave' have been suggested, but in my opinion these are no less ambiguous or cumbersome.

The procedure may require modifications for some flows where as yet unanticipated phenomena may occur, but it appears to provide a flexible and workable framework for most cases. It should also be applicable, with a little modification, to situations where the flow contracts horizontally as well as vertically, provided the two contractions occur at the same place. Conditions for critical flow under these circumstances may be found without difficulty (Armi 1986).

The author is grateful to Roger Grimshaw for discussions, to Ronald Smith for indicating the procedure of Appendix C, to Fiona Guest for computing the results of figure 6, to the hospitality of JISAO and the Department of Oceanography, University of Washington, Seattle, where this work was completed, and to Carol Drew for typing the manuscript.

Appendix A. Algorithms for computing jumps

A.1. *Free upper surface*

Using the equations and notation of §4, for hydrostatic flow we have

$$p_s - P_s = -g\rho_{n+1} \sum_{j=1}^{n} (d_j - D_j). \tag{A 1}$$

Substituting into (4.6) then gives

$$\alpha(\xi_i) = \frac{1-\rho_{n+1}/\rho_i}{1-\rho_{n+1}/\rho_{i+1}} \frac{U_{i+1}^2}{U_i^2} \alpha(\xi_{i+1}) - \frac{g}{U_i^2}\left(1-\frac{\rho_{i+1}}{\rho_i}\right) \sum_{j=i+1}^{n} \frac{\rho_j - \rho_{n+1}}{\rho_{i+1}-\rho_{n+1}} D_j \xi_j \quad (i = 1, 2, \ldots, n-1), \tag{A 2}$$

where
$$\alpha(\xi) = \frac{2\xi}{(1+\xi)(2+\xi)}. \tag{A 3}$$

Equations (A 1) and (4.6) with $i = n$ also give

$$D_f \equiv \frac{U_n^2}{g}\left(1-\frac{\rho_{n+1}}{\rho_n}\right)^{-1} \alpha(\xi_n) - \sum_{j=1}^{n} D_j \xi_j = 0. \tag{A 4}$$

One then proceeds by taking a value of ξ_n in the range $-1 < \xi_n < \infty$ and then using (A 2) to obtain successive values of $\xi_{n-1}, \xi_{n-2}, \ldots, \xi_1$. Hence D_f may be calculated as a function of ξ_n only, and the zeroes of D_f will give permissible jumps. For weak jumps, ξ_n will be close to zero.

A.2. *Rigid upper surface*

Here the procedure is essentially the same, with (A 2), (A 4) replaced by

$$\alpha(\xi_i) = \frac{\rho_{i+1} U_{i+1}^2}{\rho_i U_i^2} \alpha(\xi_{i+1}) - \frac{g}{U_i^2}\left(1-\frac{\rho_{i+1}}{\rho_i}\right) \sum_{j=i+1}^{n} D_j \xi_j, \tag{A 5}$$

and
$$D_r \equiv \sum_{j=1}^{n} D_j \xi_j = 0, \tag{A 6}$$

respectively.

The jumps also have an energy-loss criterion which must be satisfied. This criterion is

$$\tfrac{1}{2} \sum_{i=1}^{n} \rho_i D_i U_i^3 f(\xi_i) < 0, \tag{A 7}$$

where
$$f(\xi) = \frac{-\xi^3}{(1+\xi)^2 (2+\xi)}. \tag{A 8}$$

Appendix B. Algorithms for computing the effects of changes in topographic height on the flow

B.1. *Free upper surface*

Substituting (A 1) in (2.9) gives

$$v_i^2 = 1+(v_{i+1}^2-1)\frac{U_{i+1}^2}{U_i^2}\frac{1-\dfrac{\rho_{n+1}}{\rho_i}}{1-\dfrac{\rho_{n+1}}{\rho_{i+1}}}$$

$$+\frac{2g}{U_i^2}\left(1-\frac{\rho_{i+1}}{\rho_i}\right)\sum_{j=i+1}^{n}\frac{\rho_j-\rho_{n+1}}{\rho_{i+1}-\rho_{n+1}}\left(\frac{1}{v_j}-1\right)D_j \quad (i = n-1, n-2, \ldots, 1,) \qquad \text{(B 1)}$$

where

$$v_j = \frac{u_j}{U_j}. \qquad \text{(B 2)}$$

For $i = n$ we obtain

$$d_0 \equiv h-\frac{U_n^2}{2g}\frac{v_n^2-1}{1-\dfrac{\rho_{n+1}}{\rho_n}}-\sum_{j=1}^{n}D_j\left(\frac{1}{v_j}-1\right). \qquad \text{(B 3)}$$

Choosing a value of v_n enables all successive values of v_i, $i = n-1, \ldots, 1$ to be calculated, and hence h may be evaluated as a function of v_n. Starting with $b_n = 1$, where $h = 0$, v_n is increased or decreased incrementally to give positive values of h. The chain of calculations is continued until h passes through a maximum and returns to zero. This maximum height is the critical height, where some internal wave mode has zero upstream propagation velocity at the obstacle crest.

B.2. *Rigid upper surface*

Here we have essentially the same procedure, but with (B 1), (B 3) replaced by

$$v_i^2 = 1+\frac{\rho_{i+1}U_{i+1}^2}{\rho_i U_i^2}(v_{i+1}^2-1)+\frac{2g}{U_i^2}\left(1-\frac{\rho_{i+1}}{\rho_i}\right)\sum_{j=i+1}^{n}D_j\left(\frac{1}{v_j}-1\right) \quad (i = n-1, n-2, \ldots, 1). \qquad \text{(B 4)}$$

$$d_0 \equiv h = \sum_{j=1}^{n}D_j\left(1-\frac{1}{v_j}\right) \quad (i = n-1, n-2, \ldots, 1), \qquad \text{(B 5)}$$

respectively.

Appendix C. Expression for dc/da – the rate of change of linear wave speed with amplitude of columnar disturbances

We consider linear disturbances on a layered flow U_i, D_i, $i = 1, n$, with a free upper surface. Such disturbances are governed by equations (3.5)–(3.7). We may represent a disturbance in the form

$$d_i' = ad_{0i}', \quad u_i' = au_{0i}', \quad c = c_i, \qquad \text{(C 1)}$$

(omitting x and t dependence) which will constitute some particular mode, with a a measure of the amplitude and taken as small. If this disturbance propagates as a columnar disturbance mode and adds to the mean flow, the new mean flow will be

$U_i + au'_{0i}$, $D_i + ad'_{0i}$. A new small amplitude disturbance propagating on this flow may be written

$$b(d'_{0i} + ad'_{1i}), \quad b(u'_{0i} + au'_{1i}), \quad c = c_i + ac'_i, \tag{C 2}$$

where b is another small amplitude parameter. We then have $c'_i = \mathrm{d}c/\mathrm{d}a$. Substituting (C 2) into (3.5)–(3.7) we obtain, to lowest order in a,

$$-\rho_i \frac{(U_i - c_i)^2}{gD_i} d'_{1i} + \rho_i \sum_{j=1}^{i} d'_{1j} + \sum_{j=i+1}^{n} \rho_j d'_{1j} - \frac{\rho_{n+1}(U_n - c_i)^2 d'_{1n}}{\left(1 - \dfrac{\rho_{n+1}}{\rho_n}\right) gD_n}$$
$$= \rho_i \frac{(U_i - c_i)^2}{gD_i} \left| \frac{2(u'_{0i} - c'_i)}{U_i - c_i} - \frac{d'_{0i}}{D_i} \right| d'_{0i}$$
$$+ \frac{\rho_{n+1}(U_n - c_i)^2}{gD_n} \left| \frac{2(u'_{0n} - c'_i)}{U_n - c_i} - \frac{d'_{0n}}{D_n} \right| d'_{0n} \quad (i = 1, \ldots, n). \tag{C 3}$$

If we multiply this equation by d'_{0i} and sum over i, the resulting left-hand side vanishes, by virtue of (3.6) with $d'_i = d'_{0i}$. The right-hand side then gives the expression for c'_i

$$\frac{\mathrm{d}c}{\mathrm{d}a} = c'_i = -\frac{-3 \sum_{i=1}^{n} \left| \dfrac{\rho_i (U_i - c_i)^2}{D_i^2} d'^3_{0i} + \dfrac{\rho_{n+1}(U_n - c_i)^2}{D_n^2} d'^2_{0n} d'_{0i} \right|}{2 \sum_{i=1}^{n} \left| \dfrac{\rho_i (U_i - c_i)}{D_i} d'^2_{0i} + \dfrac{\rho_{n+1}(U_n - c_i)}{D_n} d'_{0n} d'_{0i} \right|}. \tag{C 4}$$

If the upper layer is rigid rather than free, the same analysis yields the same expression but with $d'_{0n} = 0$.

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